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Article

Restrictive Approximation Algorithm for Kuramoto–Sivashinsky Equation

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Abstract: A new finite difference Algorithm called the Restrictive Taylor Approximation (RTA) is implemented to find the numerical solution of Kuramoto–Sivashinsky equation which is nonlinear partial differential equation. This method is a new explicit method. The accuracy of the method is assessed in terms of the absolute error which is very close to zero. We solve also Burger's equation and Viscous Burger equation.

Keyword: Kuramoto–Sivashinsky Equation; Restrictive Taylor Approximation; Finite difference; Exponential matrix; Burger's equation.

Mathematics subject classification (2010): 30A10, 30C10, 30D15.

1. Introduction

The Kuramoto–Sivashinsky (K-S) equation was derived independently by Kuramoto [1] and Sivashinsky [2]. This equation was originally derived in the context of plasma instabilities, flame front propagation, and phase turbulence in reaction-diffusion system [3]. It occurs in context of long waves on the interface between two viscous fluids [4], unstable drift waves in plasmas, reaction-diffusion systems [5], and flame front instability [2]. The KS equation is useful to model solitary pulses in a falling thin film [6].This equation models the fluctuations of the position of a flame front, the motion of a fluid going down a vertical wall, or a spatially uniform oscillating chemical reaction in a homogeneous medium [7].

In this paper, we consider the Kuramoto-Sivashinsky (K-S) equation in the form

$$\frac{\partial \mathrm{u}}{\partial \mathrm{t}} + \mathrm{u}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \alpha\frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}^2} + \upsilon\frac{\partial^4 \mathrm{u}}{\partial \mathrm{x}^4} = 0 \tag{1}$$

where α and υ are real constants.

Many authors studied the Kuramoto–Sivashinsky equation by using various analytical and numerical techniques. The KS equation has been studied by many methods, including Chebyshev spectral collocation method [8], including orthogonal cubic spline collocation method [9], discontinuous Galerkin method [10], tanh-function method [11] and by BDF method [12]. The numerical solution of the Kuramoto–Sivashinsky equation is found by radial basis function (RBF) based on mesh-free method [13] and the Lattice Boltzmann method [14]. In this paper, numerical solution of the Kuramoto–Sivashinsky equation using the Restrictive Taylor approximation [15, 16, 17] is presented.

2. The Restrictive Taylor Approximation (RTA) Method

Constructing a function $f(x)$ that can expand in the Restrictive Taylor approximation as the form [15,16]:

$$\begin{aligned} RTA_{n,f(x)}(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) \\ + \frac{\varepsilon(x-a)^n}{n!}f^{(n)}(a) \end{aligned} \tag{2}$$

where ε is a parameter to be determined by adding the following condition.

$$RTA_{n,f}(x_a) = f(x_a) \tag{3}$$

Some points x_a in the domain of the function $f(x)$. The function $RT_{n,f}(x)$ is called restrictive Taylor approximation of order n of the function $f(x)$ at the point $x = a$. Assume the function $f(x)$ and its derivatives up to an order $n+1$ are continuous in a certain neighborhood of a point (a). Suppose, furthermore, that x is any value of the argument from the indicated neighborhood and ε is the restrictive Taylor parameter, then there is a point ξ lies between the points (a) and x such that the formula:

$$f(x) = RTA_{n,f(x)}(x) + \mathcal{R}_{n+1}(x, \varepsilon(x)) \tag{4}$$

is true, for which

$$\mathcal{R}_{n+1}\big(x,\varepsilon(x)\big)=\frac{\varepsilon(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi)-\frac{n(\varepsilon-1)^{n+1}(x-a)^{n+1}}{(n+1)!\,(x-\xi)}f^{(n)}(\xi),\quad \xi\in[a,x] \tag{5}$$

where $\mathcal{R}_{n+1}(x,1)$ is the Taylor remindar term.

2.1. Restrictive Taylor Approximation of the Exponential Matrix

The exponential matrix *exp(rA)* can be formally defined by the convergent power series

$$\exp(rA)=I+rA+\frac{r^2}{2!}A+\cdots=\sum_{n=0}^{\infty}\frac{r^n}{n!}A^n,\quad A^0=I \tag{6}$$

where A is a $(N-1)\times(N-1)$ matrix [18].

In the case of RT approximation of single function the term ϵ_i in equation (2), it can be reduced to the square restrictive matrix Γ.

where $\Gamma\ =\epsilon_i$I and I are the identity matrix.

$$\Gamma\ =\epsilon_i I=\begin{pmatrix}\epsilon_{1,j} & & \cdots & & 0 & \\ & \epsilon_{2,j} & & & & \\ \vdots & & \ddots & & \vdots & \\ & & & \epsilon_{N-2,j} & & \\ 0 & & \cdots & & & \\ & & & & & \epsilon_{N-1,j}\end{pmatrix} \tag{7}$$

So the Restrictive Taylor Approximation of The exponential matrix exp(rA) takes the form[19,20,21]:

$$RT_{1,\exp(rA)}(r)=I+r\Gamma A\ =I+r\epsilon_i I\ A \tag{8}$$

$$u_{i,j+1}=u_{i,j}+ku_t+\frac{k^2}{2!}u_{tt}+\frac{k^3}{3!}u_{ttt}+\cdots \tag{9}$$

$$u_{i,j+1}=Exp\left[k\frac{\partial}{\partial t}\right]u_{i,j} \tag{10}$$

2.2. Restrictive Taylor's Approximation for Kuramoto–Sivashinsky Equation

The exact solution of grid representation of Eq. (1) is given by

$$u_{i,j+1}=Exp\left[-k\left(u\frac{\partial}{\partial x}+\alpha\frac{\partial^2}{\partial x^2}+\upsilon\frac{\partial^4}{\partial x^4}\right)\right]u_{i,j} \tag{11}$$

$$u_{i,j+1}=Exp[-k(uD_x+\alpha D_x^2+\upsilon D_x^4)]\,u_{i,j} \tag{12}$$

The approximation of the partial derivative D_x , D_x^2 and D_x^4 at the grid point (ih,jk) will take the forms

$$D_x u = \frac{u_{i+1,j} - u_{i-1,j}}{2h} \quad (13)$$

$$D_x^2 u = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \quad (14)$$

$$D_x^4 u = \frac{u_{i+2,j} - 4u_{i+1,j} + 6u_{i,j} - 4u_{i-1,j} + u_{i-2,j}}{h^4} \quad (15)$$

where $\Delta x = h, \ \Delta t = k$

The result of making this approximation Eq. (15) will take the form

$$\underline{U}^{j+1} = \exp(rA)\,\underline{U}^j, \quad r = \frac{k}{h^4} \quad (16)$$

where

$$\underline{U}^j = (u_{1,j}, u_{2,j}, u_{3,j} \ldots u_{N-1,j})^T \quad (17)$$

So the restrictive Taylor approximation of the first order $RT_{1,\exp}(rA)$ of the exponential matrix function $exp(rA)$ will take the form.

$RT_{1,\exp}(rA) = I + r \in_{L_1} A = I + r\epsilon_i I\, A$

where A is $N-1 \times N-1$ real constant matrix, I is the identity matrix and $\in_{L_1} = [\in_{i,L_1}]$ is the diagonal matrix of the restrictive term

$$A = \begin{pmatrix} \mathrm{m} & \mathrm{d} & f & 0 & 0 & 0 & 0 & 0 \\ \mathrm{b} & \mathrm{m} & \mathrm{d} & f & 0 & 0 & 0 & 0 \\ c & \mathrm{b} & \mathrm{m} & \mathrm{d} & f & 0 & 0 & 0 \\ 0 & c & \mathrm{b} & \mathrm{m} & \mathrm{d} & f & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & c & \mathrm{b} & \mathrm{m} & \mathrm{d} & f \\ 0 & 0 & 0 & 0 & c & \mathrm{b} & \mathrm{m} & \mathrm{d} \\ 0 & 0 & 0 & 0 & 0 & c & \mathrm{b} & \mathrm{m} \end{pmatrix}_{N-1\times N-1} \quad (18)$$

where

$$\left.\begin{aligned} \mathrm{m} &= 2\mathrm{r}\epsilon_i \alpha h^2 - 6\nu r\epsilon_i \\ b &= r\,\epsilon_i\,(M\,\frac{h^2}{2} - \alpha h^2 + 4\nu) \\ c &= -r\,\epsilon_i \nu \\ d &= r\,\epsilon_i\,(-M\,\frac{h^2}{2} - \alpha h^2 + 4\nu) \end{aligned}\right\} \quad (19)$$

$$f = -r\,\epsilon_i\,\nu$$

$$\underline{U}^{j+1} = (I + r\Gamma A)\underline{U}^j = B\underline{U}^j \quad (20)$$

Then the equivalent scalar approximation of restrictive Taylor approximation for the Kuramoto–Sivashinsky equation is on scalar form

$$u_{i,j+1} = -r\left[\left(\frac{h^3}{2}\epsilon_i\big(u_{i,j}\big)\big(u_{i+1,j} - u_{i-1,j}\big) + \alpha\, h^2\epsilon_i(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) \right.\right. \\ \left.\left. + \nu\epsilon_i\big(u_{i+2,j} - 4u_{i+1,j} + 6u_{i,j} - 4u_{i-1,j} + u_{i-2,j}\big)\right)\right] + u_{i,j} \tag{21}$$

3. Stability Analysis

We use Gerschgorin's theorem [22] to examine the stability of the finite difference equation (20).

Eq.(20) the matrix B takes the form:

$$B = \begin{pmatrix} \mathrm{D} & \mathrm{d} & f & 0 & 0 & 0 & 0 & 0 \\ \mathrm{b} & \mathrm{D} & \mathrm{d} & f & 0 & 0 & 0 & 0 \\ c & \mathrm{b} & \mathrm{D} & \mathrm{d} & f & 0 & 0 & 0 \\ 0 & c & \mathrm{b} & \mathrm{D} & \mathrm{d} & f & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & c & \mathrm{b} & \mathrm{D} & \mathrm{d} & f \\ 0 & 0 & 0 & 0 & c & \mathrm{b} & \mathrm{D} & \mathrm{d} \\ 0 & 0 & 0 & 0 & 0 & c & \mathrm{b} & \mathrm{D} \end{pmatrix}_{N-1\times N-1} \tag{22}$$

where

$$\begin{aligned} \mathrm{D} &= 2\mathrm{r}\epsilon_i\alpha h^2 - 6\nu r\epsilon_i + 1 \\ b &= r\,\epsilon_i\,(M\,\frac{h^2}{2} - \alpha h^2 + 4\nu) \\ c &= -r\,\epsilon_i\nu \\ d &= r\,\epsilon_i\,(-M\,\frac{h^2}{2} - \alpha h^2 + 4\nu) \\ f &= -r\,\epsilon_i\,\nu \end{aligned} \tag{23}$$

for $\mathrm{D} > 0$

we get $\|B\|_\infty = 1$

which will ensure the stability for

$$r\epsilon_i(3\nu - \alpha h^2) < \frac{1}{2} \tag{24}$$

4. Numerical Examples

Consider the Kuramoto–Sivashinsky Eq. (1); the domain is covered by a rectangular grid with spacing h and k in the x, t directions respectively. The grid point (x, t) is denoted by (ih, jk) and $u(ih, jk) = u_{i,j}$ where $i = 0(1)\ N$; j is a non-negative integer. In this section, we solve four examples to check performance of the method. Accuracy of the results is checked by the absolute error.

Example 1: We consider the Kuramoto–Sivashinsky Eq. (1) for $\alpha = -1, \quad \upsilon = 1$ with exact solution is given by

$$u(x,t) = b + \frac{15}{19\sqrt{19}}\left[\tanh^3(k(x - bt - x_o)) - 3\tanh(k(x - bt - x_o))\right] \tag{25}$$

For $b = 5$, $k = \frac{1}{2\sqrt{19}}$, $x_0 = -25$ The boundary and the initial conditions are taken from the exact solution. Computational domain is [0,1]. The numerical results are obtained by RTA scheme Eq. (21) with h=0.1, k=0.00001. In Table 1, we show the absolute error. We expand the computation domain to [-30,30] and plot the RTA solution and exact solution in Fig. 1 for the values $\alpha = -1,\ \upsilon = 1$ with $h = 0.4$ and $k = 0.001$ at times t =1, 4, 7.

Table 1: The absolute error of the solution **Example 1.** using RTA at time step k=0.00001 and distance step h=0.1 for various values of (*t*, *x*) at $\alpha = -1, \quad \upsilon = 1$

T	Absolute Error		
	x = 0.1	x = 0.5	x = 0.9
0.0001	$8.881784197001252 \times 10^{-16}$	0	$1.77635683940025 \times 10^{-15}$
0.005	$9.769962616701378 \times 10^{-15}$	$2.753353101070388 \times 10^{-14}$	0
0.01	$9.769962616701378 \times 10^{-15}$	$2.8421709430404 \times 10^{-14}$	0
0.05	$4.440892098500626 \times 10^{-15}$	$1.77635683940025 \times 10^{-15}$	$5.329070518200751 \times 10^{-15}$
0.09	$4.440892098500626 \times 10^{-15}$	$4.529709940470639 \times 10^{-14}$	$1.509903313490213 \times 10^{-14}$
0.1	$7.993605777301127 \times 10^{-15}$	$5.684341886080801 \times 10^{-14}$	$1.687538997430238 \times 10^{-14}$
0.4	$1.740829702612245 \times 10^{-13}$	$8.748557434046234 \times 10^{-13}$	$1.820765760385256 \times 10^{-13}$
0.5	$3.028688411177427 \times 10^{-13}$	$1.500133350873511 \times 10^{-12}$	$3.055333763768431 \times 10^{-13}$
0.9	$1.858957432432362 \times 10^{-12}$	$8.982148358427366 \times 10^{-12}$	$1.77280412572145 \times 10^{-12}$
1	$2.784439345759892 \times 10^{-12}$	$1.342659317060679 \times 10^{-11}$	$2.639666263348772 \times 10^{-12}$

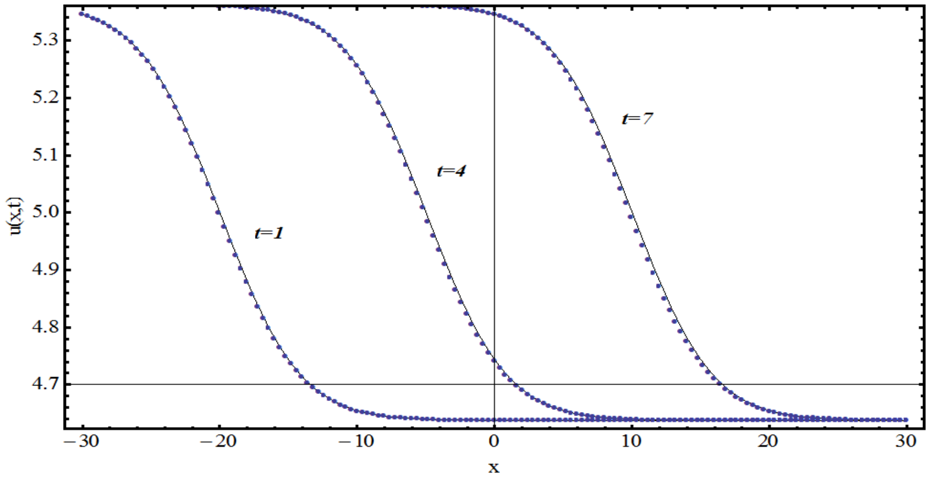

Fig. 1. RTA solutions (dote) and exact solutions (Solid) at times t = 1, 4, 7 of Example 1 for $\alpha = -1, \quad \upsilon = 1$

Example 2: We consider the Kuramoto–Sivashinsky Eq. (1) for $\alpha = 1, \quad \upsilon = 1$ with exact solution is given by

$$u(x,t) = b + \frac{15}{19}\sqrt{\frac{11}{19}}[11\tanh^3(k(x - bt - x_o)) - 9\tanh(k(x - bt - x_o))] \tag{26}$$

For $b = 5,\ k = \frac{1}{2}\sqrt{11/19}, x_0 = -12,$ The boundary and the initial conditions are taken from the exact solution. The computational domain is [0,1]. The numerical results are obtained by RTA scheme Eq. (21) at h=0.1, k=0.00001. In Table 2, we show the absolute error. We expand the computation domain to [-30,30] and plot the RTA solution and exact solution in Fig. 2 for the values $\alpha = 1, \quad \upsilon = 1$ with $h = 0.4$ and $k = 0.001$ at times t =1, 2, 3.

Table 2: The absolute error of the solution **Example 2.** using RTA at time step k=0.00001 and distance step h=0.1 for various values of (t, x) at $\alpha = 1, \quad \upsilon = 1$

T	Absolute Error		
	$x = 0.1$	$x = 0.5$	$x = 0.9$
0.0001	$2.664535259100375 \times 10^{-15}$	$8.881784197001252 \times 10^{-15}$	$2.664535259100375 \times 10^{-15}$
0.005	$5.062616992290714 \times 10^{-14}$	$1.678657213233236 \times 10^{-13}$	$4.263256414560601 \times 10^{-14}$
0.01	$1.376676550535194 \times 10^{-13}$	$5.524469770534779 \times 10^{-13}$	$1.110223024625156 \times 10^{-13}$
0.05	$1.116440273563057 \times 10^{-12}$	$4.936495656693296 \times 10^{-12}$	$9.041656312547275 \times 10^{-13}$
0.09	$2.601474591301667 \times 10^{-12}$	$1.159072837708663 \times 10^{-11}$	$2.106759211528697 \times 10^{-12}$
0.1	$3.073097332162433 \times 10^{-12}$	$1.370814572965173 \times 10^{-11}$	$2.490452288839151 \times 10^{-12}$
0.4	$7.619505026923434 \times 10^{-11}$	$3.420010941113105 \times 10^{-10}$	$6.177636180382251 \times 10^{-11}$
0.5	$1.781526037802905 \times 10^{-10}$	$7.997709161600142 \times 10^{-10}$	$1.444080410806236 \times 10^{-10}$
0.9	$4.380242835111403 \times 10^{-9}$	$1.963648532665729 \times 10^{-8}$	$3.5392107022858 \times 10^{-9}$
1	$9.624979924183208 \times 10^{-9}$	$4.310962875564428 \times 10^{-8}$	$7.764083598260641 \times 10^{-9}$

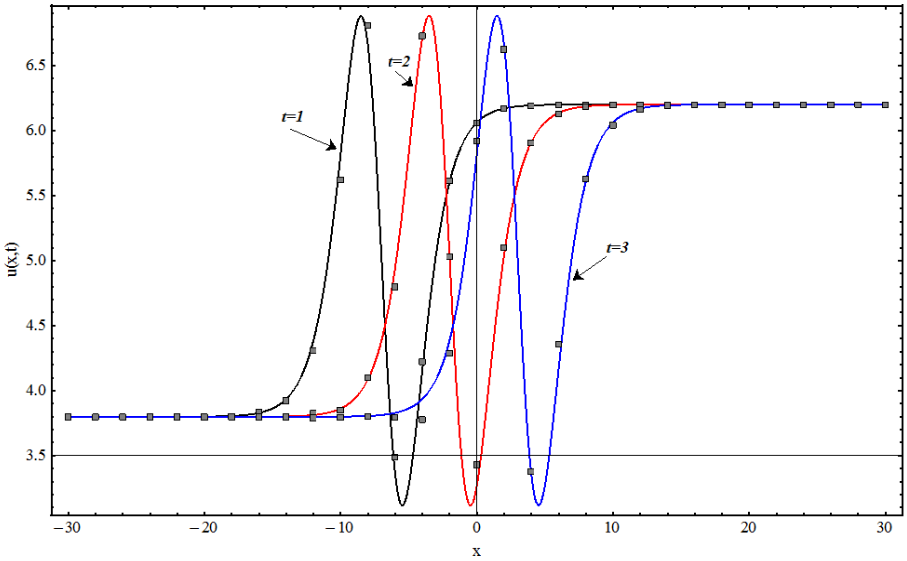

Fig. 2. RTA solutions (square) and exact solutions (Solid) at times t = 1, 2, 3 of Example 2 for $\alpha = 1, \quad \upsilon = 1$

Example 3. We consider the Eq. (1) for $\alpha = 0, \quad \upsilon = 0$ Burger with exact solution is given by

$$u(x,t) = \frac{x}{R+t} \tag{27}$$

The boundary and the initial conditions are taken from the exact solution. The computational domain is [0,1]. The numerical results is obtained by RTA scheme Eq. (21) at $h = 0.1, k = 0.001$. In Table 2, we show the absolute error.

Table 3 The absolute error of the solution **Example 3.** using RTA at time step k=0.001 and distance step h=0.1 for various values of (t, x) at $\alpha = 0, \quad \upsilon = 0 \quad \mathrm{R} = 500$

t	**Absolute Error**		
	***x* = 0.2**	***x* = 0.6**	***x* = 0.8**
0.01	$2.168404344971009 \times 10^{-19}$	$6.505213034913027 \times 10^{-19}$	$8.673617379884035 \times 10^{-19}$
0.05	$4.553649124439118 \times 10^{-18}$	$1.366094737331735 \times 10^{-17}$	$1.821459649775647 \times 10^{-17}$
0.1	$1.702197410802242 \times 10^{-17}$	$5.117434254131581 \times 10^{-17}$	$6.808789643208968 \times 10^{-17}$
0.4	$2.596664203102783 \times 10^{-16}$	$7.778066385411009 \times 10^{-16}$	$1.038665681241113 \times 10^{-15}$
0.9	$1.299578934049749 \times 10^{-15}$	$3.899658373995862 \times 10^{-15}$	$5.198315736198999 \times 10^{-15}$
1	$1.602450810933575 \times 10^{-15}$	$4.808219794538715 \times 10^{-15}$	$6.409803243734302 \times 10^{-15}$
2	$6.354671563263414 \times 10^{-15}$	$1.906179207533664 \times 10^{-14}$	$2.541760205088117 \times 10^{-14}$
5	$3.900981100646294 \times 10^{-14}$	$1.170181573167949 \times 10^{-13}$	$1.560095368863256 \times 10^{-13}$
7	$7.562375205216743 \times 10^{-14}$	$2.268610646560809 \times 10^{-13}$	$3.023870216722901 \times 10^{-13}$
10	$1.518759076835074 \times 10^{-13}$	$4.556218683587909 \times 10^{-13}$	$6.070595415241797 \times 10^{-13}$

5. Results and Conclusion

The Restrictive Taylor Approximation (RTA) is a type of finite difference approximation. The results in **Table 1** are very close to exact solution and we can observe that at some points the absolute error is zero. This proves that our restrictive method at some cases obtains the exact solution. The executive time of calculating of Restrictive Taylor Approximation is relatively very small, the executive time of calculating 10000 steps is 0.43 seconds, this is relatively very small. RTA gives the numerical solution which is very close to the exact solution if it is known at one level of time, for example at $t = k, i.e. u(x,t) = u(ih,k) \;\; i = 1(1)N$. Without knowing the exact solution at one level, we try to use an approximate, fast efficient and accurate method with suitable very small step sizes h and k, to get the needed almost exact solution at specific level, after which we continue the usual Restrictive Taylor process. All mathematical calculations and graphs are done using *mathematica 8*.

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